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AN ALGORITHM FOR AVERAGE COSTS DENUMERABLE STATE SEMI-MARKOV DECISION PROBLEMS WITH APPLICATIONS TO CONTROLLED PRODUCTION AND QUEUEING SYSTEMS

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An algorithm for average costs denumerable state semi-Markov decision problems with applications to controlled production and queueing systems*)

by

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ABSTRACT

This paper presents a computational approach for typical applications of average costs denumerable state semi-Markov decision problems as arising in controlled production and queueing systems. This approach combines policy-iteration and embedding techniques to develop, by exploiting the structure of the application considered, a tailor-made algorithm for computing an optimal policy within a given class of intuitively reasonable policies having a simple form. We consider as applications an M/G/1 queueing system with controllable service time distribution, a discrete production problem in which the inventory is controlled by turning on or off the production and an M/M/c queueing system in which the number of servers operating can be controlled.

KEY WORDS & PHRASES: *Semi-Markov decision problems, denumerable state space, algorithm, embedding techniques, applications, M/G/1 queue with controllable service time, inventory control in discrete production system, M/M/c queue with variable number of servers.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In the last fifteen years there has been a considerable interest in the study of optimal design and control of production and queueing systems, cf. the bibliography by Crabill et al (1977) and the survey papers by Prabhu and Stidham (1974) and Sobel (1974). On the one hand the literature deals with the static analysis of an intuitively reasonable control rule having a simple form and on the other hand a large number of papers are concerned with verifying the optimality of such a simple control rule among a larger class of control rules. However, so far little attention has been paid to the development of computationally tractable algorithms for the numerical solution of the above control problems which usually involve an unbounded number of states.

In this paper we shall present for structured applications of *average costs denumerable state* semi-Markov decision problems a computational approach which appeared to be quite successfull for the applications considered. This approach combines policy-iteration and embedding techniques in such a way that we need only to perform calculations for a *finite* number of states without having truncated the unbounded state space. For a specific application we exploit its *structure* and usean *embedding* technique to develop a tailor-made policy-iteration algorithm which deals only with a finite embedded set of states in any iteration and generates a sequences of improved policies having the desired simple form. In each of the applications considered the dimension of the embedded sets of states is considerably smaller than that of any appropiately chosen finite state space approximation and this dimensionality reduction is important in view of computations.

In section 2 we shall present the embedding approach for the average costs denumerable state semi-Markov decision model and in section 3 we shall give three applications. The first application considers an M/G/1 queueing system in which the queue size can be controlled by varying the service time distribution where fixed switching-cost are assumed. The second application deals with a production-inventory system in which discrete production occurs and the inventory is controlled by turning off or on the production facility where a start-up time is assumed.

The third application considers an M/M/c queueing system with a variable number of servers where servers can be turned on or off at switching-costs.

We note that the embedding approach is also very useful in solving stochastic control problems in which the system can be continuously controlled and the decision processes are represented by controlled Markov drift processes involving compound Poisson processes.Typical applications arise in controlled dam-storage and production-inventory problems in which the inventory can be continuously controlled by varying the release and production rate respectively. These control problems in which the times and costs incurred between two successive decisions depend on the whole control rule used are not covered by the semi-Markov decision model. For this type of control problems a general Markov decision approach involving embedding techniques was first developed by De Leve (1964) and subsequently studied in De Leve et al. (1970, 1977), cf. also Tijms (1976a, 1977) and Tijms and Van Der Duyn Schouten (1978). We finally note that the above control problems could also be analysed by the technique of diffusion process approximations introduced by Bather (1966, 1968) and further studied by Chernoff and Petkau (1977), Faddy (1974), Puterman (1976), Rath (1977) and Whitt (1973).

2. THE EMBEDDING APPROACH

We are concerned with a dynamic system which at decision epochs beginning with epoch 0 is observed and classified into one of the states of a denumerable state space I. After observing the state of the system, a decision must be taken where for any state $i \in I$ a finite set of possible actions is available. If at a decision epoch action a is taken in state i, then the time until the next decision epoch and the state at the next decision epoch are random with a known joint probability distribution function which only depends on the last observed state i and the subsequently chosen action a. We further assume that a cost structure is imposed on the model in the following way. If action a is chosen in state i, then an immediate fixed cost is incurred and in addition until the next decision epoch the evolution of the system can be described by some stochastic process in which we incur costs (e.g. a cost rate and fixed costs) in a well-defined way where the cost evolution is only determined by the last observed state i and the subsequently chosen action a. For ease the costs are assumed to be non-negative. We note that such a detailed cost structure is typical for applications.

We now define the following familiar quantities. Given that at epoch 0 the system is in state $i \in I$ and action $a \in A(i)$ is chosen, define

$p_{ij}(a)$ = probability that at the next decision epoch the state will be j.

$\tau(i,a)$ = unconditional expected transition time until the next decision epoch.

$c(i,a)$ = expected costs incurred until the next decision epoch.

We assume that $\inf_{i,a} \tau(i,a) > 0$. We take the *long-run average expected costs per unit time* as optimality criterion. We confine ourselves to a finite subclass F_0 of the class of all stationary policies where a stationary policy to be denoted by f^∞ is a control rule which always prescribes the single action $f(i) \in A(i)$ whenever the system is observed in state i. In applications the class F_0 will typically consist of policies having a simple form so that we know or may reasonably expect that this class contains a policy which is average cost optimal within the class of all policies. Anyhow we shall not be concerned with the verification of such an optimality result but we shall only focus on the computation of the best policy within the class F_0. We make the following assumption.

ASSUMPTION 1. *For any* $f^\infty \in F_0$ *there is a state* $s_f \in I$ *such that for any* $i \in I$ *the quantities* $T(i,f)$ *and* $K(i,f)$ *are finite where*

$T(i,f)[K(i,f)]$ = *total expected time [total expected costs incurred] until the next decision epoch at which a transition occurs into state* s_f *given that the initial state is* i *and policy* f^∞ *is used.*

Now we first derive some preliminary results before discussing the embedding approach. We first observe that, by assumption 1 and $\inf_{i,a} \tau(i,a) > 0$, the expected number of transitions until the first return to state s_f is finite for any initial state i when using policy $f^\infty \in F_0$. Consequently for any $f^\infty \in F_0$ there is a unique stationary probability distribution $\{\pi_j(f), j \in I\}$ such that for all $i,j \in I$

$$(2.1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{k}(f) = \pi_j(f), \quad \pi_j(f) = \sum_{i\in I} p_{ij}(f(i))\, \pi_i(f),$$

where $P^n(f) = (p_{ij}^n(f))$ is the n-fold matrix product of the stochastic matrix $P(f) = (p_{ij}(f(i)))$ with itself, cf. Chung (1960). For any $f^\infty \in F_0$ define

$$(2.2) \qquad g(f) = \sum_{j\in I} c(j,f(j))\pi_j(f) \Big/ \sum_{j\in I} \tau(j,f(j))\pi_j(f).$$

Denote by $Z(t)$ the total costs incurred in $[0,t)$ and, for $n = 0,1,\ldots,$ let T_n be the nth decision epoch and let C_n be the total costs incurred in $[T_n, T_{n+1})$. Then, by the proof of Theorem 7.5 in Ross (1970) and the ergodic Theorem on p. 89 in Chung (1960) we find for any $f^\infty \in F_0$ that for initial state $i = s_f$

$$(2.3) \qquad \lim_{t\to\infty} E_{i,f^\infty} \frac{[Z(t)]}{t} = \frac{K(s_f,f)}{T(s_f,f)} = \lim_{n\to\infty} \frac{E_{i,f^\infty}[\sum_{k=0}^{n} C_k]}{E_{i,f^\infty}[\sum_{k=0}^{n}(T_{k+1} - T_k)]} = g(f),$$

where E_{i,f^∞} denotes the expectation when the initial state is i and policy f^∞ is used. Further, using (2.1) and assumption 1, we have that (2.3) holds for any $i \in I$ where moreover, with probability 1, $Z(t)/t$ converges to $g(f)$ as $t \to \infty$. Hence under policy f^∞ the long-run average (expected) cost equals $g(f)$ independent of the initial state. For any $f^\infty \in F_0$, define now the relative cost function $w(i,f)$, $i \in I$ by

$$(2.4) \qquad w(i,f) = K(i,f) - g(f)T(i,f) \qquad \text{for all } i \in I.$$

Observe that, by (2.3)-(2.4), for any $f^\infty \in F_0$

$$(2.5) \qquad w(s_f,f) = 0.$$

We now introduce the following assumption.

<u>ASSUMPTION 2</u>. *For any* $f^\infty \in F_0$, $\sum_{j\in I} p_{ij}(a)w(j,f)$ *and* $\sum_{j\in I}\pi_j(\overline{f})w(j,f)$ *converge absolutely for any* $i \in I_0$, $a \in A(i)$ *and* $\overline{f}^\infty \in F_0$ *where* I_0 *is the finite set of states defined by* $I_0 = \{i \in I \mid f(i) \neq g(i)$ *for some* $f^\infty, g^\infty \in F_0\}$.

For any $f^\infty \in F_0$ define the "policy-improvement" quantity $T(i,a,f)$, $i \in I_0$ and $a \in A(i)$ by

$$(2.6) \qquad T(i,a,f) = c(i,a)-g(f)\tau(i,a)+\sum_{j\in I} p_{ij}(a)w(j,f).$$

We have the following familiar results (cf. De Leve et al (1977) and Derman and Veinott (1967)).

<u>THEOREM 2.1</u>. (a) *Suppose assumption* 1 *hold. Then, for any* $f^\infty \in F_0$,

$$(2.7) \qquad w(i,f) = c(i,f(i))-g(f)\tau(i,f(i))+\sum_{j\in I} p_{ij}(f(i))w(j,f), \qquad i \in I$$

where $\sum_j p_{ij}(f(i))w(j,f)$ *converges absolutely for any* $i \in I$.
(b) *Suppose assumptions* 1-2 *hold. If for some* $f^\infty, \overline{f}^\infty \in F_0$,

$$(2.8) \qquad T(i,\overline{f}(i),f) \leq w(i,f) \text{ for all } i \in I_0,$$

then $g(\overline{f}) \leq g(f)$ *where the strict inequality sign holds if in* (2.8) *the strict inequality sign holds for some state* i *which is positive recurrent under the stochastic matrix* $P(\overline{f})$. *The assertion remains true when the inequality signs are reversed.*
(c) *Suppose assumptions* 1-2 *hold. If for some* $f_0^\infty \in F_0$,

$$(2.9) \qquad \min_{a\in A(i)} T(i,a,f_0) = w(i,f_0) \text{ for all } i \in I_0,$$

then $g(f_0) \leq g(f)$ *for all* $f^\infty \in F_0$, *i.e. policy* f_0^∞ *is average cost optimal within the class* F_0.

PROOF (a) Note that $K(i,f) = c(i,f(i))+\sum_{j\neq s_f} p_{ij}(f(i))K(j,f)$ for all $i \in I$. A similar relation applies to $T(i,f)$. Together these relations, (2.4)- (2.5) and the nonnegativity of $K(i,f)$ and $T(i,f)$ imply part (a).
(b) Since $\overline{f}(i) = f(i)$ for all $i \notin I_0$, we have for all $i \in I$

$$c(i,\overline{f}(i))-g(f)\tau(i,\overline{f}(i))+\sum_{j\in I} p_{ij}(\overline{f}(i))w(j,f) \leq w(i,f).$$

Multiplying both sides of this inequality by $\pi_i(\overline{f})$, summing over i and using (2.1) and the fact that $\pi_i(\overline{f}) > 0$ for i positive recurrent under $P(\overline{f})$, we get part (b).

(c) This part is an immediate consequence of part (b).

Now, by part (b) of Theorem 1 and the fact that $T(i,f(i),f) = w(i,f)$ for any $i \in I_0$, we can always design a policy-iteration scheme which generates a sequence of improved policies within the class F_0. However, in practice we can only apply such an iteration scheme if for any policy $f^\infty \in F_0$ we can numerically evaluate the finite number of quantities

(2.10) $\qquad g(f)$, $w(i,f)$ and $\sum_{j\in I} p_{ij}(a)w(j,f)$, $i \in I_0$ and $a \in A(i)$.

In general we cannot numerically evaluate $g(f)$ and $w(i,f)$, $i \in I_0$ by directly solving the infinite system of linear equations (2.7). We shall now demonstrate how the quantities in (2.10) could be computed in specific applications by using an embedding technique and exploiting the structure of the problem considered.

For any policy $f^\infty \in F_0$, choose a *finite* set A_f with $s_f \in A_f$. Fix now policy $f^\infty \in F_0$. Consider the embedded Markov chain giving the state of the system at the decision epochs at which the system assumes a state in the embedded set A_f when policy f^∞ is used. For this embedded Markov chain, define the following one-step transition probabilities, one-step expected transition times and one-step expected costs,

$\tilde{p}_{ij}(f)$ = the probability that the system will assume state j at the decision epoch at which the first return to the set A_f occurs when the initial state is $i \in I$, $j \in A_f$.

$\tilde{\tau}(i,f)$ = the total expected time until the decision epoch at which the first return to the set A_f occurs when the initial state is $i \in I$.

$\tilde{c}(i,f)$ = the total expected costs incurred until the decision epoch at which the first return to the set A_f occurs when the initial state is $i \in I$.

Observe that, by $s_f \in A_f$ and assumption 1, $\sum_{j\in A_f} \tilde{p}_{ij}(f) = 1$ and the quantities $\tilde{\tau}(i,f)$ and $\tilde{c}(i,f)$ are finite for any $i \in I$. We are now in a position to prove the main result of this section.

THEOREM 2.2. *Suppose that assumption* 1 *holds. Then for any* $f^\infty \in F_0$ *the finite system of linear equations in* $\{g, v(i),\ i \in A_f\}$,

$$(2.11)\qquad v(i) = \tilde{c}(i,f) - g\tilde{\tau}(i,f) + \sum_{j\in A_f} \tilde{p}_{ij}(f)v(j) \ \textit{for}\ i \in A_f$$

$$(2.12)\qquad v(s_f) = 0$$

has the unique solution $g = g(f)$, $v(i) = w(i,f)$, $i \in A_f$. *Further*,

$$(2.13)\qquad w(i,f) = \tilde{c}(i,f) - g(f)\tilde{\tau}(i,f) + \sum_{j\in A_f} \tilde{p}_{ij}(f)w(j,f) \ \textit{for all}\ i \in I.$$

PROOF. Fix $f^\infty \in F_0$. Since the stochastic matrix $P(f)$ has no two disjoint closed sets, the finite stochastic matrix $(\tilde{p}_{ij}(f))$, $i,j \in A_f$ has also no two disjoint closed sets, cf. Theorem 2.2 in Federgruen et al (1978). Now, by a well-known result in Markov decision theory, the system of linear equations (2.11)-(2.12) has a unique solution. Hence, by (2.5) and (2.12), it suffices to verify (2.13). To do this, observe that, by $s_f \in A_f$,

$$K(i,f) = \tilde{c}(i,f) + \sum_{\substack{j\in A_f \\ j\neq s_f}} \tilde{p}_{ij}(f)K(j,f) \text{ for all } i \in I.$$

A similar relation applies to $T(i,f)$, $i \in I$. Using these relations and (2.4)-(2.5), we get (2.13).

We now return to the problem of the numerical evaluation of the quantities in (2.10) for a given policy $f^\infty \in F_0$. In specific applications it is often possible to choose the set A_f in such a way that, by exploiting the structure of the application considered, analytical expressions (or simple recursion formulae) can be obtained for the quantities $\tilde{c}(i,f)$, $\tilde{\tau}(i,f)$ and $\tilde{p}_{ij}(f)$ for all $i \in I$ and $j \in A_f$. Then we first compute the numbers $g(f)$ and $w(i,f)$, $i \in A_f$ by solving the finite system of linear equations (2.11)-(2.12).

Next, using (2.13), we can compute w(i,f) and $\Sigma_j p_{ij}(a)w(j,f)$ for any $i \in I_0$ and $a \in A(i)$. The order of the system of linear equations to be solved in the value-determination step of the policy-iteration algorithm is equal to the dimension of the set A_f when f^∞ is the current policy. In the applications considered the dimension of the embedded set A_f turned out to be considerably smaller than that of any appropriately chosen approximating finite state space. As already pointed out, by the flexibility of the policy-improvement step, the policy-iteration algorithm can always be designed in such a way that a sequence of improved policies within the class F_0 will be generated. It seems to be difficult to give conditions under which the policy-iteration algorithm will converge to a policy $f_0^\infty \in F_0$ for which the optimality condition (2.9) holds so that policy f_0^∞ is at least average cost optimal within the class F_0. In each of the applications considered the policy-iteration method generated within the finite class F_0 a sequence of strictly improved policies so that after a finite number of iterations convergence happened to a policy f_0^∞ (say) for which we could numerically verify the optimality condition (2.9) in all examples tested. We found the well-known empirical phenomenon of the very fast convergence of the policy-iteration algorithm; in the examples tested the number of iterations varied between 3 and 15. We conclude this section by remarking that in controlled production and queueing systems involving switching-costs the verification of the optimality of a simple policy within a larger class of policies is an extremely difficult theoretical problem for which still no satisfactory theory has been developed. Although we have not obtained any theoretical optimality result for the applications considered, it is our conjecture that for each of these applications the developed tailor-made policy-iteration algorithm may be a fruitful tool for the verification of the optimality of a simple policy within a larger class of policies.

3. APPLICATIONS TO CONTROLLED PRODUCTION AND QUEUEING SYSTEMS

Before discussing the applications, we first give for the standard M/G/1 queue some known results that will be frequently used hereafter. For completeness we include a simple and instructive derivation of these results, cf. also Hordijk and Tijms (1976).

3.1. *Some busy period results for the* M/G/1 *queue.*

Consider a single server system where customers arrive in accordance with a Poisson process with rate λ and the service times of the customers are independent, non-negative random variables each being distributed as the random variable S with $\lambda ES < 1$ and $ES^2 < \infty$. Upon arrival a customer is immediately served if the server is idle and he waits in line if the server is busy. Given that at epoch 0 a service starts when $i \geq 1$ customers are present, define the following random variables

ζ_i = total amount of time spent by customers in the system during the first service.

τ_i = the first epoch at which the system becomes empty.

ν_i = the number of customers arriving in $(0,\tau_i)$.

W_i = total amount of time spent by customers in the system during $(0,\tau_i)$.

LEMMA 3.1 *For any* $i \geq 1$

$$E\zeta_i = iES+\tfrac{1}{2}\lambda ES^2,\quad E\tau_i = \frac{iES}{1-\lambda ES},\quad E\nu_i = \frac{i\lambda ES}{1-\lambda ES}, \tag{3.1}$$

$$EW_i = \tfrac{1}{2}i(i-1)E\tau_1+i\left\{\frac{ES}{1-\lambda ES} + \frac{\lambda ES^2}{2(1-\lambda ES)^2}\right\} \tag{3.2}$$

PROOF. A well-known property of the Poisson process states that under the condition of n arrival epochs in (0,s), each of these n arrival epochs is uniformly distributed on (0,s), cf. Theorem 2.3. in Ross (1970). Denote by S_1 the length of the first service and let N_1 be the number of arrivals during the first service. Then $E(\zeta_i \mid S_1 = s, N_1 = n) = is+ns/2$ which gives $E\zeta_i$. To prove the other relations, note that the distributions of τ_i, ν_i and W_i are independent of the order in which the customers are served and note also that any customer in fact generates a busy period. By these standard arguments from queueing theory, we have for all $i \geq 1$

$$E\tau_i = iE\tau_1,\quad E\nu_i = iE\nu_1 \text{ and } EW_i = \tfrac{1}{2}i(i-1)E\tau_1+iEW_1.$$

Hence it suffices to verify (3.1)-(3.2) for i = 1. We have

$E(\tau_1 | S_1 = s, N_1 = n) = s + nE\tau_1$, $E(W_1 | S_1 = s, N_1 = n) = s + \frac{ns}{2} +$
$+ \frac{1}{2} n(n-1)E\tau_1 + nEW_1$.

By unconditioning we get $E\tau_1$ and EW_1. The proof is completed by noting that, by Wald's equation, $1 + E\nu_1 ES = E\tau_1$.

REMARK 3.1.1. In the above we have assumed that the customers arrive one at a time. Consider now the case of batch arrivals at epochs generated by a Poisson process with rate λ where the batch sizes are independent positive random variables having a common discrete probability distribution with mean β. Assuming that $\lambda\beta ES < 1$ and customers are served one at a time, a trivial modification of the above proof shows that the relations (3.1)-(3.2) remain true provided that we replace λ by $\lambda\beta$.

We now discuss the first application.

3.2. *An M/G/1 queueing system with controllable service time distribution.*

Consider a single server system where customers arrive in accordance with a Poisson process with rate λ. Each customer is served by using one of two available service types $k = 1,2$. At any service completion epoch the server has to decide which service type to use for the next service. The service time of a customer has probability distribution function F_k when service type k is used where $F_k(0) < 1$ for $k = 1,2$. It is assumed that $F_2(t) \geq F_1(t)$ for all $t \geq 0$ so that service type 2 is "faster" than service type 1. Denote by μ_k the first moment of F_k and for $j \geq 2$ denote by $\mu_k^{(j)}$ the j^{th} moment of F_k. We assume that $\lambda\mu_2 < 1$, $\mu_1^{(2)} < \infty$ and $\mu_2^{(3)} < \infty$. The following costs are incurred. There is a holding cost at rate $h.i$ when i customers are present and a service cost at rate r_k when the server is busy and uses service type k. Further, a fixed switching-cost of R_k is incurred when at a service completion epoch the server decides to switch from service type k to the other one. The cost parameters are assumed to be nonnegative.

This controlled queueing problem can be represented by a semi-Markov decision model in which the decision epochs are given by the service completion epochs and at any decision epoch the system can be classified into one of the states of the denumerable state space

$$I = \{i \mid i = 0,1,\ldots\} \cup \{i' \mid i = 0,1,\ldots\}$$

where state $i(i')$ corresponds to the situation in which the number of customers present is i and service type 1(2) was used for the service just completed. For any state $s \in I$ the set of available actions is given by $A(s) = \{1,2\}$ where action k prescribes to use service type k for the next service. If action k is taken in state s, the time until the next decision epoch is distributed as the service time under service type k if $s \neq 0,0'$ and is distributed as the sum of the time between successive arrivals and the service time under service type k otherwise. Hence, for $k = 1,2$,

$$\text{(3.3)} \qquad \tau(i,k) = \tau(i',k) = \mu_k \text{ for } i \geq 1 \text{ and } \tau(0,k) = \tau(0',k) = \frac{1}{\lambda} + \mu_k.$$

If action k is chosen in state s, we incur as immediate cost the appropiate switching-cost if any and until the next decision epoch the evolution of the system can be described by the queue length process given service type k where costs at a rate of $h.j+r_k\delta(j)$ are incurred when j customers are in the system with $\delta(0) = 0$ and $\delta(j) = 1$ for $j \geq 1$. Using (3.1), we have for $i \geq 0$

$$\text{(3.4)} \qquad c(i,1) = h(i \vee 1)\mu_1 + \tfrac{1}{2}h\lambda\mu_1^{(2)} + r_1\mu_1, \quad c(i',2) = h(i \vee 1)\mu_2 + \tfrac{1}{2}h\lambda\mu_2^{(2)} + r_2\mu_2,$$

$$\text{(3.5)} \qquad c(i,2) = R_1 + c(i',2), \quad c(i',1) = R_2 + c(i,1),$$

where $i \vee 1 = \max(i,1)$. For $k=1,2$, define

$$\text{(3.6)} \qquad p_k(j) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF_k(t), \quad j = 0,1,\ldots,$$

i.e. $p_k(j)$ is the probability that j customers arrive during a service time under service type k. The transition probabilities $p_{st}(k)$ can be directly expressed in terms of the probabilities $p_k(j)$ and for reasons of space we omit these obvious expressions.

For a given positive integer N, denote by $F_0 = F_0^{(N)}$ the class of stationary policies having the following simple form. Any policy $f^\infty \in F_0$ is characterized by two switch-over levels i_1 and i_2 with $0 \leq i_2 \leq i_1 < N$ and

$i_1 \geq 1$. Under this policy to be denoted by $f^\infty = (i_1, i_2)$ the server switches from service type 1 to service type 2 only at the service completion epochs where the queue size is larger than i_1 and the server switches only from service type 2 to service 1 at the service completion epochs where the queue size is less than or equal to i_2. It is intuitively reasonable to expect that there exists an average cost optimal policy which belongs to the class F_0 with N sufficiently large. For the case of no switching-costs this optimality question was studied by Crabill (1972), Gallisch (1977) and Tijms (1976b). However, we wish only to compute the best policy within the class F_0. To do this we first note that, by choosing $s_f = i_2'$ for any policy $f^\infty = (i_1, i_2)$, assumption 1 of section2 is satisfied. In fact we shall see below that the functions $T(s,f)$ and $K(s,f)$, $s \in I$ are bounded by a linear and quadratic function of s respectively. Without giving details, we note that by this result and the assumption of stochastically ordered service times with $\mu_2^{(3)} < \infty$ it can also be shown that assumption 2 of section 2 is satisfied. For any $f^\infty = (i_1, i_2) \in F_0$, we choose

$$A_f = \{i \mid i = 0,\dots,i_1\} \cup \{i_2'\}.$$

We shall now demonstrate that for this choice of A_f analytical expressions can be given for the quantities $\tilde{c}(s,f)$, $\tilde{\tau}(s,f)$ and $\tilde{p}_{st}(f)$. Fix policy $f^\infty = (i_1, i_2)$. Observing that for initial state i' with $i > i_2$ the first entry state in A_f is state i_2' we find by Lemma 3.1

(3.7) $$\tilde{c}(i',f) = h(i_2 E\tau_{i-i_2} + EW_{i-i_2}) + r_2 E\tau_{i-i_2} \quad \text{for } i > i_2$$

where $E\tau_j$ and EW_j are given by (3.1) and (3.2) in which $ES = \mu_2$ and $ES^2 = \mu_2^{(2)}$. Further

(3.8) $$\tilde{c}(i,f) = R_1 + \tilde{c}(i',f) \text{ for } i > i_1, \quad \tilde{c}(i',f) = R_2 + \tilde{c}(i,f) \text{ for } 0 \leq i \leq i_2$$

$$\tilde{c}(i,f) = c(i,1) + \sum_{j=i_1-i+2}^{\infty} \tilde{c}(i-1+j,f)p_1(j) \text{ for } 1 \leq i \leq i_1,$$

$$\tilde{c}(0,f) = \tilde{c}(1,f).$$

The formula for $\tilde{c}(i,f)$, $1 \leq i \leq i_1$ can be simplified. Therefore we introduce the following useful shorthand notation. For any $m \geq i_2-1$, $1 \leq i \leq m+2$ and $k = 1,2$, define

$$H(i,m,i_2,k,\alpha,\beta,\gamma) = \sum_{j=m-i+2}^{\infty} [\alpha ET_{i-1+j-i_2}+\beta EW_{i-1+j-i_2}+\gamma]p_k(j) \tag{3.9}$$

where α,β,γ are given constants. We find after some algebra

$$H(i,m,i_2,k,\alpha,\beta,\gamma) = [\bar{\alpha}.(i-1)^2+\bar{\beta}.(i-1)+\bar{\gamma}][1-\sum_{j=0}^{m-i+1} p_k(j)]+$$
$$+[2\bar{\alpha}.(i-1)+\bar{\beta}.][\lambda\mu_k-\sum_{j=0}^{m-i+1} jp_k(j)]+\bar{\alpha}[\lambda^2\mu_k^{(2)}+\lambda\mu_k-\sum_{j=0}^{m-i+1} j^2p_k(j)],$$

where

$$\bar{\alpha} = \frac{\beta\mu_2}{1-\lambda\mu_2}, \quad \bar{\beta} = \frac{1}{1-\lambda\mu_2}\{\alpha\mu_2+\tfrac{1}{2}\beta\mu_2-\beta\mu_2 i_2+\frac{\beta\lambda\mu_2^{(2)}}{2(1-\lambda\mu_2)}\}, \quad \bar{\gamma} = -\bar{\alpha}i_2^2-\bar{\beta}i_2+\gamma.$$

Using (3.1), (3.2), (3.4) and (3.7) - (3.9), we find for $1 \leq i \leq i_1$,

$$\tilde{c}(i,f) = hi\mu_1+\tfrac{1}{2}h\lambda\mu_1^{(2)}+r_1\mu_1+H(i,i_1,i_2,1,\ r_2+hi_2,h,R_1). \tag{3.10}$$

The formulae for $\tilde{\tau}(i,f)$ and $\tilde{\tau}(i',f)$ for $i \geq 1$ follow by putting $r_1 = r_2 = 1$, $h = R_1 = R_2 = 0$ in the corresponding formulae for $\tilde{c}(i,f)$ and $\tilde{c}(i',f)$, $i \geq 1$. Clearly

$$\tilde{\tau}(0,f) = \tilde{\tau}(0',f) = \frac{1}{\lambda} + \tilde{\tau}(1,f). \tag{3.11}$$

Further, it follows directly that

$\tilde{p}_{i'i_2'}(f) = 1$ for $i > i_2$, $\tilde{p}_{ii_2'}(f) = 1$ for $i > i_1$,

$\tilde{p}_{it}(f) = p_1(t-i+1)$ for $1 \leq i \leq i_1$, $i-1 \leq t \leq i_1$, $\tilde{p}_{ii_2'}(f)=1-\sum_{j=0}^{i_1-i+1} p_1(j)$

for $1 \leq i \leq i_1$,

$\tilde{p}_{i't}(f) = \tilde{p}_{it}(f)$ for $0 \leq i \leq i_2$, $\tilde{p}_{0t}(f) = \tilde{p}_{1t}(f)$.

Hence we see that the functions $\tilde{c}(s,f)$, $\tilde{\tau}(s,f)$ and $\tilde{p}_{st}(f)$ depend on f in a simple way and can be easily computed. We next specify the system of linear equations (2.11)-(2.12). Therefore we first observe that, by (2.7) and (3.8),

$$w(i',f) = c(i',1)-g(f)\tau(i',1)+\sum_{j=0}^{\infty} p_1(j)w(i-\delta(i)+j,f) = \tag{3.12}$$

$$= R_2+w(i,f) \qquad \text{for } 0 \le i \le i_2.$$

Further, by (2.7), (3.8) and (3.11), we find

$$w(0,f) = \frac{-g(f)}{\lambda} +w(1,f). \tag{3.13}$$

By (3.10), (3.12)-(3.13) and $s_f = i_2'$, the linear equations (2.11)-(2.12) reduce to

$$v(0)+ \frac{g}{\lambda} -v(1) = 0 \tag{3.14}$$

$$v(i)+g\{\mu_1+H(i,i_1,i_2,1,1,0,0)\} - \sum_{j=0}^{i_1-i+1} p_1(j)v(i-1+j) = \tag{3.15}$$

$$= hi\mu_1+\tfrac{1}{2}h\lambda\mu_1^{(2)}+r_1\mu_1+H(i,i_1,i_2,1,r_2+hi_2,h,R_1) \text{ for } 1 \le i \le i_1$$

$$v(i_2) = -R_2. \tag{3.16}$$

This system of linear equations can be very efficiently solved. Successively for $i = i_1,\dots,1$ we can express $v(i-1)$ as a linear combination of g and $v(i_1)$ by using the equation (3.15) for $v(i)$. Next, by using the two equations (3.14) and (3.16), we can solve for g and $v(i_1)$. Hence, by (3.15), $v(i) = \alpha(i)v(i_1)+\beta(i)g+\gamma(i)$ for $0 \le i \le i_1$ where $\alpha(i)$, $\beta(i)$, $\gamma(i)$ can be successively computed for $i = i_1,\dots,0$. We have $\alpha(i_1) = 1$ and

$$\alpha(i-1) = p_1(0)^{-1}\{\alpha(i)- \sum_{j=1}^{i_1-i+1} p_1(j)\alpha(i-1+j)\} \text{ for } i = i_1,\dots,1.$$

Similar recursions apply to $\beta(i)$ and $\gamma(i)$. We next specify $w(s,f)$ for $s \notin A_f$.

For $0 \le i \le i_2$ we have that $w(i',f)$ is given by (3.12). Using (2.13), (3.7) and $w(i_2',f) = 0$, we find

(3.17) $$w(i',f) = hEW_{i-i_2} + \{r_2+hi_2-g(f)\}E\tau_{i-i_2} \text{ for } i > i_2$$

where $E\tau_j$ and EW_j are given by (3.1)-(3.2) with $ES = \mu_2$ and $ES^2 = \mu_2^{(2)}$. Finally, by (2.7),

(3.18) $$w(i,f) = R_1+w(i',f) \text{ for } i > i_1.$$

We next specify the test quantity $T(s,a,f)$. By (2.6)-(2.7) and (3.5),

$$T(i',1,f) = c(i',1)-g(f)\tau(i',1)+\sum_{j=0}^{\infty} p_1(j)w(i-1+j,f) =$$

$$= \begin{cases} R_2+w(i,f) & \text{for } i_2 < i \le i_1 \\ R_2+T(i,1,f) & \text{for } i > i_1. \end{cases}$$

Using (3.4), (3.9), (3.12) and (3.17), we get from (2.6) that

$$T(i',2,f) = hi\mu_2+\tfrac{1}{2}h\lambda\mu_2^{(2)}+r_2\mu_2-g(f)\mu_2+\sum_{j=0}^{i_2-i+1} p_2(j)\{R_2+w(i-1+j,f)\}+$$

$$+H(i,i_2,i_2,2,r_2+hi_2-g(f),h,0) \text{ for } 1 \le i \le i_2.$$

Using (3.4), (3.9) and (3.17)-(3.18), we get from (2.6) that

$$T(i,1,f) = hi\mu_1+\tfrac{1}{2}h\lambda\mu_1^{(2)}+r_1\mu_1-g(f)\mu_1+\{1-\delta(i-i_1-1)\}p_1(0)w(i_1,f)+$$

$$+H(i,i-1-\delta(i-i_1-1),i_2,1,r_2+hi_2-g(f),h,R_1) \text{ for } i > i_1,$$

where $\delta(j) = 1$ for $j \ge 1$ and $\delta(0) = 0$. Finally, by (2.6)-(2.7) and (3.5),

$$T(i,2,f) = \begin{cases} R_1+T(i',2,f) & \text{for } 1 \le i \le i_2 \\ R_1+w(i',f) & \text{for } i_2 < i \le i_1 \end{cases}$$

We shall now describe the algorithm. Choose an integer N and a policy $f^\infty = (i_1,i_2)$ with $0 \le i_2 \le i_1 < N$ and $i_1 \ge 1$.

Algorithm

Step 1 (*value-determination step*). Let $f^\infty = (i_1,i_2)$ be the current policy. Compute from (3.14)-(3.16) the numbers $g(f)$ and $w(i,f)$, $i = 0,\dots,i_1$.

Step 2 (*policy-improvement step*). (a) First determine an integer $\bar{i}_2$ with $0 \le \bar{i}_2 \le i_1$. Define $\bar{i}_2$ as the largest integer k such that $i_2 < k \le i_1$ and $T(i',1,f) < w(i',f)$ for all $i_2 < i \le k$ if such an integer k exists, otherwise let $\bar{i}_2$ be equal to $\ell-1$ with ℓ the smallest integer such that $1 \le \ell \le i_2$ and $T(i',2,f) < w(i',f)$ for all $\ell \le i \le i_2$ if such an integer ℓ exists, and otherwise let $\bar{i}_2 = i_2$.

(b) Next determine an integer $\bar{i}_1$ with $\bar{i}_2 \le \bar{i}_1 < N$ and $\bar{i}_1 \ge 1$. Define $\bar{i}_1$ as the largest integer k such that $i_1+1 \le k < N$ and $T(i,1,f) < w(i,f)$ for all $i_1+1 \le i \le k$ if such an integer k exists, otherwise let $\bar{i}_1$ be equal to $\ell-1$ with ℓ the smallest integer such that $\bar{i}_2 < \ell \le i_1$, $\ell \ge 2$ and $T(i,2,f) < w(i,f)$ for all $\ell \le i \le i_1$ if such an integer ℓ exists, and otherwise let $\bar{i}_1 = i_1$.

Step 3 Let $\bar{f}^\infty = (\bar{i}_1,\bar{i}_2)$. If $\bar{f}^\infty = f^\infty$, then stop, otherwise go to step 1 with the previous policy $f^\infty = (i_1,i_2)$ replaced by the new policy $\bar{f}^\infty = (\bar{i}_1,\bar{i}_2)$.

The algorithm stops after a finite number of iterations since the class F_0 is finite and $g(\bar{f}) < g(f)$ if the new policy $\bar{f}^\infty = (\bar{i}_1,\bar{i}_2)$ is not equal to the previous policy $f^\infty = (i_1,i_2)$. This follows from Theorem 2.1(b) by observing that the states $\bar{i}_1$ and $\bar{i}_2'$ are positive recurrent under policy $\bar{f}^\infty$. In all examples tested we could numerically verify the condition (2.9) for the finally obtained policy f_0^∞ (say) so that this policy is at least average cost optimal within the class $F_0 = F_0^{(N)}$. We note that the chosen integer N only appears in step 2(b) of the algorithm. It gives no difficulties to enlarge N during the algorithm if desired. It needs hardly to be said that the embedding approach results in a policy iteration algorithm which compares quite favourably with any computational approach for an approximating finite state space model since the dimension of an approximating finite state space will usually much larger than that of the embedded set A_f for which besides the system of linear equations (3.14)-(3.16) can be very efficiently solved.

Consider the following numerical example with constant service times.

$\lambda = 1,\ \mu_1 = 1,\ \mu_2 = 0.8,\ h = 0.02,\ r_1 = 2,\ r_2 = 50.$

In table 3.1 we give for various values of the switching-cost the (i_1,i_2) policies generated by the algorithm where n and $g(i_1,i_2)$ denote the iteration number and the average costs. We have chosen N = 200 and (i_1=100, i_2=0) as starting policy. In the two examples in table 3.1 an optimal policy was found after 8 and 6 iterations respectively.

	$R_1 = R_2 = 0$		$R_1 = R_2 = 50$	
n	(i_1,i_2) ;	$g(i_1,i_2)$	(i_1,i_2) ;	$g(i_1,i_2)$
1	(100,0) ;	4.49718	(100,0) ;	4.50654
2	(122,100) ;	3.98023	(122,100) ;	3.99908
3	(82,82) ;	3.97213	(114,78) ;	3.97869
4	(96,82) ;	3.95903	(109,84) ;	3.97847
5	(97,96) ;	3.95357	(110,82) ;	3.97789
6	(94,94) ;	3.95328	(111,81) ;	3.97781
7	(95,94) ;	3.95327		
8	(95,95) ;	3.95325		

Table 3.1 The iterations

REMARK 3.1

(a) The analysis can be rather straightforwardly extended to the case where more than two service types are available. However, in view of results in Faddy (1974) and Sobel (1974), we might expect that in many situations the fairly complicated control rules involving more than two service types are not significantly better than the bang-bang control rules involving only the slowest and the fastest service type.

(b) The analysis applies also to the case where customers arrive in batches, cf. remark 3.1. Further, the holding cost rate may be allowed to depend on the service type used where in fact a nonlinear holding cost rate may be assumed when the fastest service type is not used. The analysis also carries on if the arrival rate depends on the service type used. However, for a controllable arrival rate it might be desirable to include the epochs of arrivals occurring when service type 1 is used as extra decision epochs.

If the service time distribution F_1 is exponential, this can be done with only obvious modifications of the analysis.

3.3 *A discrete production-inventory problem with start-up times.*

Consider a production system which operates only intermittently to manufacture a single product. Production is stopped if the inventory is sufficiently high and production is restarted if the inventory has dropped sufficiently low. Demands for the product occur at epochs generated by a Poisson process with rate λ where demand in excess of stock is backordered. For ease of presentation we will assume that the demand size is equal to one unit (cf. remark 3.3 below for the case of a general discrete demand distribution). The units of the product are manufactured one at a time where the time to manufacture one unit is distributed as the positive random variable T_p with $\lambda ET_p < 1$ and $ET_p^2 < \infty$. After the completion of the production of one unit, either a new production is started or the production facility is shut-down. At any demand epoch occurring when the system is shut-down, the production facility can be reactivated where it takes a start-up time distributed as the nonnegative random variable T_s with $ET_s < \infty$ before the next production actually starts. We assume an upper bound U ($\leq \infty$) for the number of units that can be kept in stock. The following costs are considered. There is holding cost of $h > 0$ per unit kept in stock per unit time and for any unit backordered there is a fixed backorder cost $\pi_1 \geq 0$ and a linear backorder cost $\pi_2 > 0$ per unit time the backorder exists. There is an operating cost at rate $r_1 \geq 0$, $r_2 \geq 0$ and $r_3 \geq 0$ when the production facility is producing, shut-down and being reactivated respectively. Finally, a fixed set-up cost of $R \geq 0$ is incurred when the production system is reactivated.

This production problem can be represented by a semi-Markov decision model in which the decision epochs are given by the production completion epochs and the demand epochs occurring when the system is shut-down. At any decision epoch the system can be classified into one of the states of the denumerable state space

$$I = \{i \mid i \leq U\} \cup \{i' \mid i \leq U\},$$

where state i corresponds to the situation in which the stock on hand minus on backorder equals i and a production has been just completed and state i' corresponds to the situation in which the stock on hand minus on backorder equals i and a demand has just occurred when the system is shut-down. For any state $s \in I$, two possible actions $a = 0$ and $a = 1$ are available. In state i the action $a = 0$ ($a = 1$) means to stop production (to continue production) and in state i' the action $a = 0$ ($a = 1$) means to keep the system shut-down (to reactivate the system). We clearly have

$$(3.19) \qquad c(i,0) = c(i',0) = \begin{cases} (hi+r_2)/\lambda & \text{for } i \geq 1 \\ -\pi_2 i/\lambda+\pi_1+r_2/\lambda & \text{for } i \leq 0. \end{cases}$$

To give the formula for $c(s,1)$, denote by F_p and F_s the probability distribution functions of T_p and T_s respectively and let the random variable A_k denote the epoch at which the demand for the kth unit occurs. Observe that A_k has a gamma distribution with parameters (k,λ). For $k = 0,1,\ldots,$ let

$$p_k = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^k}{k!}\, dF_p(t) \text{ and } q_k = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^k}{k!}\, dF_s(t),$$

i.e. $\{p_k\}$ and $\{q_k\}$ are the probability distributions of the cumulative demand in a production time and start-up time respectively. Using the well-known fact that given n demand epochs have occurred in $(0,t)$ each of these demand epochs is uniformly distributed on $(0,t)$, we find

$$(3.20) \qquad c(i,1) = \begin{cases} -\pi_2 iET_p+\pi_2\lambda ET_p^2/2+\pi_1\lambda ET_p+r_1ET_p, & i < 0 \\ h\sum_{k=1}^{i} E\min(T_p,A_k)+r_1ET_p+ \\ \qquad +\int_0^\infty dF_p(t) \sum_{n=i+1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(n-i)(\pi_1+\pi_2 t/2), & i \geq 0. \end{cases}$$

After some routine algebra, we find for $i \geq 0$

$$(3.21)\quad \begin{aligned} c(i,1) &= \frac{h}{\lambda}\sum_{k=1}^{i}\{k-\sum_{j=0}^{k}(k-j)p_j\}+\pi_1\{\lambda ET_p-\sum_{n=0}^{i} np_n-i+i\sum_{n=0}^{i} p_n\}+ \\ &+\frac{\pi_2}{2\lambda}\{\lambda^2 ET_p^2-\sum_{n=0}^{i+1} n(n-1)p_n-i\lambda ET_p+i\sum_{n=0}^{i+1} np_n\}+r_1 ET_p. \end{aligned}$$

In view of computations it is important to note that $c(i,1)$ for $i \geq 0$ can be recursively computed. For any i denote by $c_q(i)$ the expression obtained from $c(i,1)$ by replacing T_p, $p_n (n \geq 1)$ and r_1 by T_s, $q_n (n \geq 1)$ and r_3 respectively. Then we have

$$(3.22)\quad c(i',1) = R+c_q(i)+\sum_{k=0}^{\infty} c(i-k,1)q_k \text{ for all } i,$$

which expression can be further simplified by substituting the formula for $c(i,1)$, $i < 0$. The formula for $\tau(s,a)$ clearly follows by putting $r_1 = r_2=r_3 = 1$ and $h = \pi_1 = \pi_2 = R = 0$ in the corresponding formula for $c(s,a)$. The transition probabilities $p_{st}(a)$ can be easily expressed in the probability distributions $\{p_k\}$ and $\{q_k\}$. For reasons of space we omit these obvious expressions.

Before defining the class F_0, we introduce the following shorthand notations. Let

$$a = \frac{\pi_2 ET_p}{2(1-\lambda ET_p)},\quad b = -a-\frac{1}{1-\lambda ET_p}\{\frac{\pi_2\lambda ET_p^2}{2(1-\lambda ET_p)}+\pi_1\lambda ET_p+r_1 ET_p\},$$

$$c = \frac{-ET_p}{1-\lambda ET_p},$$

and for $i = 0,1,\ldots,$ let

$$(3.23)\quad P_1(i) = \sum_{k=i+2}^{\infty}\{a(i+1-k)^2+b(i+1-k)\}p_k,\quad P_2(i) = c\sum_{k=i+2}^{\infty}(i+1-k)p_k$$

$$(3.24)\quad Q_1(i) = \sum_{k=i+1}^{\infty}\{a(i-k)^2+b(i-k)\}q_k,\quad Q_2(i) = c\sum_{k=i+1}^{\infty}(i-k)q_k.$$

We have

$$(3.25)\qquad \begin{aligned} P_1(i) &= \{a(i+1)^2+b(i+1)\}\{1-\sum_{k=0}^{i+1} p_k\}-\{2a(i+1)+b\}\{\lambda ET_p-\sum_{k=0}^{i+1} kp_k\}+ \\ &+a\{\lambda^2ET_p^2+\lambda ET_p-\sum_{k=0}^{i+1} k^2p_k\}. \end{aligned}$$

Similar expressions apply to $P_2(i)$, $Q_1(i)$ and $Q_2(i)$.

For a given integer N with $0 \le N \le U$, denote by $F_0 = F_0^{(N)}$ the finite class of stationary policies having the following simple form. Any policy $f^\infty \in F_0$ is characterized by two integers m and M with $-N < m \le M < N$ and under this policy to be denoted by $f^\infty = (m,M)$ the inventory level (= stock on hand minus stock on backorder) is controllled as follows. If the inventory level is larger than M, do not produce until the inventory level drops to or below m and at that time reactivate the production facility and continue production until the inventory level becomes M+1. The optimality of such a simple control rule within a larger class of control rules has been studied by Sobel (1970). However, we only focus on the computation of the best policy within the class F_0 of simple policies. To avoid an overburdened notation, we consider only policies $f^\infty = (m,M) \in F_0$ with $M \ge 0$. For any policy $f^\infty = (m,M) \in F_0$, choose $s_f = m'$ and

$$A_f = \{i \mid i = 0,\dots,M\} \cup \{m'\}.$$

Fix now $f^\infty = (m,M)$. To give an explicit expression for $\tilde{c}(i,f)$, $i < 0$, we observe that under the (m,M) policy the inventory process when the system is producing can be described by the queue length process in the M/G/1 queue where the service time is distributed as the production time T_p. Together this observation and the relations (3.1)-(3.2) with S replaced by T_p imply

$$(3.26)\qquad \tilde{c}(i,f) = \pi_2 EW_{-i}+\pi_1 E\nu_{-i}+r_1 E\tau_{-i} = ai^2+bi \text{ for } i < 0.$$

Next, using (3.23) and (3.26), we find

$$\tilde{c}(i,f) = c(i,1)+\sum_{k=i+2}^{\infty} p_k\, \tilde{c}(i+1-k,f)+\delta_{iM}p_0\, \tilde{c}(M+1,f) =$$

$$= c(i,1)+P_1(i)+\delta_{iM}p_0\ \tilde{c}(M+1,f) \quad \text{for } 0 \le i \le M,$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. Further, we find

$$\tilde{c}(i',f)=\begin{cases} h\{i(i+1)-m(m+1)\}/2\lambda+r_2(i-m)/\lambda & \text{for } i > m \text{ if } m \ge 0 \\ \{hi(i+1)+\pi_2 m(m+1)\}/2\lambda-\pi_1 m+r_2(i-m)/\lambda & \text{for } i \ge 1 \text{ if } m < 0 \\ \pi_2\{m(m+1)-i(i+1)\}/2\lambda-\pi_1(m-i)+r_2(i-m)/\lambda & \text{for } m < i \le 0 \text{ if } m < 0. \end{cases}$$

$$\tilde{c}(i',f) = c(i',1)+\sum_{k=0}^{\infty} q_k\ \tilde{c}(i-k,f) =$$

$$= c(i',1)+\sum_{k=0}^{i} q_k\ \tilde{c}(i-k,f)+Q_1(i) \quad \text{for } i \le m,$$

$$\tilde{c}(i,f) = \tilde{c}(i',f) \quad \text{for } i > M.$$

The formula for $\tilde{\tau}(s,f)$ follows by putting $r_1 = r_2 = r_3 = 1$ and $h = \pi_1 = \pi_2 = R=0$ in the corresponding one for $\tilde{c}(s,f)$. Explicit expressions for the transition probabilities $\tilde{p}_{st}(f)$ can also be easily derived. We shall not give these expressions apart since they will appear in the system of linear equations to be specified below. Before specifying (2.11)-(2.12), we first relate $w(i,f)$ and $w(i',f)$ for $i \le m$. By (2.7) and (3.22),

$$w(i',f) = c(i',1)-g(f)\tau(i',1)+\sum_{k=0}^{\infty} q_k \sum_{j=0}^{\infty} p_j w(i-k+1-j,f) =$$

$$(3.27)\qquad = R+c_q(i)-g(f)ET_s+\sum_{k=0}^{\infty} q_k\{c(i-k,1)-g(f)\tau(i-k,1)+\sum_{j=0}^{\infty} p_j w(i-k+1-j,f)\}=$$

$$= R + c_q(i)-g(f)ET_s+\sum_{k=0}^{\infty} q_k w(i-k,f) \quad \text{for } i \le m.$$

We further note that, by (2.13),

$$(3.28)\qquad w(i,f)=\tilde{c}(i,f)-g(f)\tilde{\tau}(i,f)+w(0,f) \quad \text{for } i < 0.$$

Hence, using (3.24), (3.26) and (3.28), we have

$$(3.29)\qquad w(i',f) = R+c_q(i)-g(f)ET_s+\sum_{k=0}^{i} q_k w(i-k,f)+$$

$$+Q_1(i)-g(f)Q_2(i)+(1-\sum_{k=0}^{i} q_k)w(0,f) \text{ for } i \le m.$$

By $w(m',f) = 0$ and (3.29), the linear equations (2.11)-(2.12) become

$$(3.30) \qquad v(i)+g\tilde{\tau}(i,f)-\sum_{k=\delta_{iM}}^{i} p_k v(i+1-k)-(1-\sum_{k=0}^{i} p_k)v(0) = \tilde{c}(i,f),\ 0 \le i \le M$$

$$(3.31) \qquad g\{ET_s+Q_2(m)\}-\sum_{k=0}^{m} q_k v(m-k)-(1-\sum_{k=0}^{m} q_k)v(0) = R+c_q(m)+Q_1(m).$$

This system of linear equations can also be very efficiently solved. Successively for $i = 0,\dots, M-1$ we can express $v(i+1)$ as a linear combination of $v(0)$ and g by using equation (3.30) for $v(i)$. Then, using the equation (3.30) for $v(M)$ and equation (3.31), we can solve for g and $v(0)$ and next compute $v(1),\dots,v(M)$. We now specify the other $w(s,f)$. Next to the relations (3.28)-(3.29), we have

$$w(i',f) = \tilde{c}(i',f)-g(f)\tilde{\tau}(i',f),\ i > m \text{ and } w(i,f) = w(i',f),\ i > M.$$

We next specify the test quantity $T(s,a,f)$. Similarly to (3.27), we derive from (2.6), (2.7), (3.22), (3.24), (3.26) and (3.28) that, for all $m < i \le M$

$$T(i',1,f) = c(i',1)-g(f)\tau(i',1)+\sum_{k=0}^{\infty} q_k \sum_{j=0}^{\infty} p_j w(i-k+1-j,f) =$$

$$= R+c_q(i)-g(f)ET_s+\sum_{k=0}^{\infty} q_k w(i-k,f) =$$

$$= R+c_q(i)+Q_1(i)-g(f)\{ET_s+Q_2(i)\}+\sum_{k=0}^{i} q_k w(i-k,f)+(1-\sum_{k=0}^{i} q_k)w(0,f)$$

and, for all $i > M$,

$$T(i',1,f) = R+c_q(i)-g(f)ET_s+\sum_{k=0}^{i-M-1} q_k T(i-k,1,f)+\sum_{k=i-M}^{\infty} q_k w(i-k,f)$$

which expression can be further simplified by using (3.26) and (3.28). Further by (2.6)-(2.7), (3.28), (3.26) and (3.23), we find

$$T(i',0,f) = c(i',0)-g(f)\tau(i',0)+w((i-1)',f) \text{ for } i \le m,$$

$$T(i,1,f) = c(i,1)-g(f)\tau(i,1)+\sum_{j=0}^{i+1} p_j\, w(i+1-j,f)+P_1(i)-g(f)P_2(i)+$$
$$+(1-\sum_{j=0}^{i+1} p_j)w(0,f) \text{ for } i > M,$$

$$T(i,0,f) = \begin{cases} w(i',f) & \text{for } m < i \le M \\ T(i',0,f) & \text{for } i \le m. \end{cases}$$

We can now describe the algorithm. Choose an integer N and a policy $f^\infty = (m,M)$ with $-N < m \le M < N$ and $M \ge 0$.

Algorithm

Step 1 (value-determination step). Let $f^\infty = (m,M)$ be the current policy. Compute from (3.30)-(3.31) the numbers $g(f)$ and $w(i,f)$, $i = 0,\dots,M$.

Step 2 (policy-improvement step). (a) First determine an integer $\overline{m}$ with $-N < \overline{m} \le M$. Define $\overline{m}$ as the largest integer k such that $m < k \le M$ and $T(i',1,f) < w(i',f)$ for all $m < i \le k$ if such an integer k exists, otherwise let $\overline{m}$ be equal to $\ell-1$ with ℓ the smallest integer such that $-N+1 < \ell \le m$ and $T(i',0,f) < w(i',f)$ for all $\ell \le i \le m$ if such an integer ℓ exists, and otherwise let $\overline{m} = m$.
(b) Next determine an integer M with $\overline{m} \le \overline{M} < N$ and $\overline{M} \ge 0$. Define $\overline{M}$ as the largest integer k such that $M+1 \le k < N$ and $T(i,1,f) < w(i,f)$ for all $M+1 \le$ $\le i \le k$ if such an integer k exists, otherwise let $\overline{M}$ be equal to $\ell-1$ with ℓ the smallest integer such that $\overline{m} < \ell \le M$, $\ell \ge 1$ and $T(i,0,f) < w(i,f)$ for all $\ell \le i \le M$ if such an integer ℓ exists, and otherwise let $\overline{M} = M$.

Step 3. Let $\overline{f}^\infty = (\overline{m},\overline{M})$. If $\overline{f}^\infty = f^\infty$ then stop, otherwise go to step 1 with the previous policy f^∞ replaced by the new policy $\overline{f}^\infty$.

This algorithm stops after a finite number of iterations since the class F_0 is finite and $g(\overline{f}) < g(f)$ if the new policy $\overline{f}^\infty$ is not equal to the previous policy f^∞. In all examples tested we found that the finally obtained policy was average cost optimal within the class F_0.

Consider the following numerical example in which both T_p and T_s are deterministic.

$T_p = 0.1$, $h = 0.05$, $\pi_1 = 25$, $\pi_2 = 2.5$, $r_1 = r_2 = 0$, $r_3 = 100$, $R = 0$.

In table 3.2 we give for various values of λ and ET_s an optimal policy (m^*,M^*) with g^* its average costs and n^* the number of iterations after which this policy was found. We have chosen $N = 300$ and $(m = 150, M = 150)$ as starting policy for the algorithm. Observe from these examples that the minimal average cost is not a monotonic function in λ.

	$T_s = 0$			$T_s = 2$		
λ	(m^*,M^*)	g^*	(n^*)	(m^*,M^*)	g^*	(n^*)
8.5	(23,23)	1.1726	(6)	(32,118)	5.8151	(10)
9	(33,33)	1.6893	(9)	(39,110)	5.3281	(10)
9.5	(61,61)	3.1113	(10)	(62,114)	5.2948	(9)
9.75	(112,112)	5.6669	(11)	(109,153)	6.8447	(10)
9.9	(249,249)	12.5070	(13)	(242,283)	12.9911	(9)

Table 3.2 Optimal policies

REMARK 3.3. Consider now the case in which any demand has a discrete probability distribution $\{r_k, k \geq 1\}$ where we assume that $\lambda\beta ET_p < 1$ with β is the average demand size. For any policy $f^\infty = (m,M)$ with $M \geq 0$, choose now $s_f = M$ and $A_f = \{i | i = 0,\ldots,M\}$. Using remark 3.1, an examination of the above analysis shows that only some technical modifications are required. In particular the formula for $w(i',f)$, $i > m$ will now involve the renewal function associated with the demand distribution.

3.3 *An M/M/c queueing system with a variable number of servers.*

We consider an M/M/c queueing system in which the number of servers operating can be adjusted both at arrival and service completion epochs. The customers arrive in accordance with a Poisson process with rate λ and there are c independent servers available each having an exponentially distributed service time with mean $1/\mu$ where $\lambda/c\mu < 1$. The cost structure includes a holding cost of $h > 0$ per customer in the system per unit time, an operating cost of $w > 0$ per server turned on per unit time and a switch-over cost of $K(a,b)$ when the number of servers turned on is adjusted from a to b where $K(a,a) = 0$ and

$$K(a,b) = \begin{cases} K^+ + k^+ . (b-a) & \text{for } b > a \\ K^- + k^- . (a-b) & \text{for } b < a \end{cases}$$

with $K^+, K^-, k^+, k^- \geq 0$.

This control problem can be represented by a semi-Markov decision model in which the decision epochs are given by the arrival and service completion epochs and at any decision epoch the system can be classified into one of the states of the denumerable state space

$$I = \{(i,s) | i = 0,1,\ldots; s = 0,\ldots,c\}$$

where state (i,s) corresponds to the situation in which i customers are present and s servers are turned on. For any state the set of possible actions consists of the actions $a = 0,\ldots,c$ where action a prescribes to adjust the number of servers turned on to a. The formulae for the one-step transition probabilities and one-step expected transition times and costs are obvious and omitted for reasons of space.

For a given integer N, denote by $F_0 = F_0^{(N)}$ the finite class of stationary policies having the following simple form. Any policy $f^\infty \in F_0$ is characterized by integers $s(i)$, $S(i)$, $t(i)$ and $T(i)$ for $i = 0,1,\ldots$ such that
(a) $-1 \leq s(i) < S(i) \leq T(i) < t(i) \leq c+1$ for $i \geq 0$ where $s(N) = c-1$ and $t(N) = c+1$,
(b) $s(i) \leq s(i+1)$ and $t(i) \leq t(i+1)$ for $i \geq 0$.
If there are i customers present and s servers turned on at a decision epoch, then under this policy $f^\infty = (s(i), S(i), T(i), t(i))$ the number of servers on is adjusted upward to $S(i)$ when $s \leq s(i)$, is kept unaltered when $s(i) < s < t(i)$ and is adjusted downward to $T(i)$ when $s \geq t(i)$. Observe that under any policy in the class F_0 all c servers are turned on or left on when N or more customers are present. It is still an unproven conjecture that for N sufficiently large the class $F_0^{(N)}$ contains a policy which is average cost

optimal within the class of all possible policies, cf. Robin (1976) and Sobel (1974). However, we will only deal with the computation of a policy with minimal average cost within the class F_0. We note that as in Yadin and Naor (1967) an expression for the average costs under a given set of parameters s(i), S(i), T(i) and t(i) can be derived by steady-state analysis but the numerical evaluation of the optimal set of parameters from this expression is extremely difficult. We shall now briefly outline how to develop by the embedding approach a computationally tractable algorithm for computing the best policy within the class F_0.

For any policy $f^\infty = (s(i), S(i), T(i), t(i)) \in F_0$, define i_f as the smallest $i \le N$ with $s(i) = c-1$ and $t(i) = c+1$, i.e. under policy f^∞ all c servers are turned on or left on when i_f or more customers are present. We choose now $s_f = (i_f, c)$ and

$$A_f = \{(i,s)\,|\,s = 0,\dots,s(i) \text{ and } s = t(i),\dots,c;\ i = 1,\dots,i_f-1\} \cup$$

$$\cup\ \{(0,s)\,|\,s = 0,\dots,c\} \cup \{(i_f,s)\,|\,s = 0,\dots,c\}.$$

For this choice of A_f we can easily derive analytical expressions for the quantities $\tilde{c}(t,f)$, $\tilde{\tau}(t,f)$ and $\tilde{p}_{tu}(f)$ for $t \in I$ and $u \in A_f$. These analytical expressions are obtained by solving second-order linear difference equations. To illustrate this, choose any (i,s) with $0 < i < i_f$ and $s(i) < s < t(i)$ and let L be the largest integer such that $0 \le L < i$ and $(L,s) \in A_f$ and let R be the smallest integer such that $i < R \le i_f$ and $(R,s) \in A_f$. Then $\tilde{c}((j,s),f)$ for $L < j < R$ is given by the solution of the second-order linear difference equation

$$x(j) = \frac{1}{\lambda+\mu\min(j,s)}\{hj+ws+\lambda x(j+1)+\mu\min(j,s)x(j-1)\},\quad L < j < R$$

where $x(L) = x(R) = 0$. We omit here further details and refer to the similar equations (3.1), (3.2), (3.7), (3.8) and (3.10) in De Leve et al (1977) where this queueing problem has been analysed by using a related but more complex embedding approach. For given policy $f^\infty = (s(i), S(i), T(i), t(i)) \in F_0$ we have, by the choice $s_f = (i_f, c)$ and the following relation for $i = 0,\dots, i_f$

$$w((i,s),f) = \begin{cases} K^{+}+k^{+}.(S(i)-s)+w((i,S(i)),f), & s \le s(i) \\ K^{-}+k^{-}.(s-T(i))+w((i,T(i)),f), & s \ge t(i), \end{cases}$$

that the system of linear equations (2.11)-(2.12) can be reduced to a system of $2i_f+t(0)-s(0)-3$ linear equations in the unknowns g, v((i,S(i))), u((i,T(i))) for $0 < i < i_f$ and v((0,s)) for $s(0) < s < t(0)$. We omit the obvious details. Once these linear equations have been solved, we can compute any w((i,s),f) and T((i,s),a,f).

To explain the policy-improvement step of the algorithm to be stated below, we note that for any $0 \le i < i_f$ and $s = 0,\dots,c$

$$T((i,s),a,f) = K(s,a)+w((i,a),f) \text{ for } s(i) < a < t(i).$$

Now, by the structure of the switch-over costs, we have for any fixed i with $0 \le i < i_f$ that the smallest integer $\overline{S}(i)$ (say) which minimizes K(0,a)+ +w((i,a),f) for $s(i) < a < t(i)$ is less than or equal to the largest integer $\overline{T}(i)$ (say) which minimizes K(c,a)+w((i,a),f) for $s(i) < a < t(i)$. Further, for any $s \le s(i)$ the integer $\overline{S}(i)$ minimizes T((i,s),a,f) for $s(i) < a < t(i)$ and for any $s \ge t(i)$ the integer $\overline{T}(i)$ minimizes T((i,s),a,f) for $s(i) < a < t(i)$.

We now describe the algorithm. Let N be a given integer and choose some policy $f^{\infty} \in \mathcal{F}_0$.

Algorithm

Step 1 (value-determination step). Let $f^{\infty} = (s(i), S(i), T(i), t(i))$ be the current policy. Compute the numbers g(f), w((i,S(i)),f), w((i,T(i)),f) for $0 < i < i_f$ and w((0,s),f) for $s(0) < s < t(0)$ by solving the above described system of linear equations.

Step 2 (policy-improvement step). (a) For any $i = 0,\dots,i_f-1$ determine $\overline{S}(i)$ as the smallest integer which minimizes K(0,a)+w((i,a),f) for $s(i) < a < t(i)$ and determine $\overline{T}(i)$ as the largest integer which minimizes K(c,a)+ +w((i,a),f) for $s(i) < a < t(i)$.
(b) Successively for $i=0,\dots,N-1$, determine $\overline{s}(i)$ and $\overline{t}(i)$ as follows where we put $s(-1) = -1$ and $t(-1) = 0$.

Define $\bar{s}(i)$ as the largest integer k such that $s(i)+1 \le k \le \min[\bar{S}(i)-1, s(i+1)]$ and $K(s,\bar{S}(i))+w((i,\bar{S}(i)),f) < w((i,s),f)$ for all $s(i) < s \le k$ if such an integer k exists, otherwise let $\bar{s}(i)$ be equal to $\ell-1$ with ℓ the smallest integer such that $\min[0,s(i-1)+1] \le \ell \le s(i)$ and $T((i,s),s,f) \le \le K(s,\bar{S}(i))+w((i,\bar{S}(i)),f)$ for all $\ell \le s \le s(i)$ if such an integer ℓ exists, otherwise let $\bar{s}(i) = s(i)$. Define $\bar{t}(i)$ as the smallest integer k such that $\max[\bar{T}(i)+1, t(i-1)] \le k < t(i)$ and $K(s,\bar{T}(i))+w((i,\bar{T}(i)),f) < w((i,s),f)$ for all $k \le s < t(i)$ if such an integer k exists, otherwise let $\bar{t}(i)$ be equal $\ell+1$ with ℓ the largest integer such that $t(i) \le \ell \le \min[t(i+1)-1,c]$ and $T((i,s),s,f) \le K(s,\bar{T}(i))+w((i,\bar{T}(i)),f)$ for all $t(i) \le s \le \ell$ if such an integer ℓ exists, otherwise let $\bar{t}(i) = t(i)$.

Step 3. Let $\bar{f}^\infty = (\bar{s}(i), \bar{S}(i), \bar{T}(i), \bar{t}(i))$. If $\bar{f}^\infty = f^\infty$ then stop, otherwise go to step 1 with f^∞ replaced by $\bar{f}^\infty$.

We conclude with a numerical example in which

$$\lambda = 9.5,\ c = 10,\ \mu = 1,\ h = 10,\ w = 100,\ k^+ = k^- = 50.$$

In table 3.3 we give for various values of K^+ and K^- an optimal policy $(s^*(i), S^*(i), T^*(i), t^*(i))$ and the minimal average cost g^*.

	$K^+=K^-=0$, $g^*=1240.14$				$K^+=K^-=75$, $g^*=1247.67$			
i	$s^*(i)$	$S^*(i)$	$T^*(i)$	$t^*(i)$	$s^*(i)$	$S^*(i)$	$T^*(i)$	$t^*(i)$
0	-1	0	6	7	-1	0	7	9
1	0	1	6	7	-1	0	7	9
2	1	2	6	7	-1	0	7	9
3	1	2	7	8	-1	0	7	9
4	2	3	7	8	-1	0	8	10
5	3	4	8	9	-1	0	8	10
6	4	5	8	9	-1	0	10	11
7	4	5	9	10	2	7	10	11
8	5	6	10	11	2	7	10	11
9	6	7	10	11	4	8	10	11
10	6	7	10	11	6	8	10	11
11	7	8	10	11	6	8	10	11
12	7	8	10	11	6	10	10	11
13	8	9	10	11	6	10	10	11
14	9	10	10	11	7	10	10	11
15					8	10	10	11
16					9	10	10	11

Table 3.3. Optimal policies

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